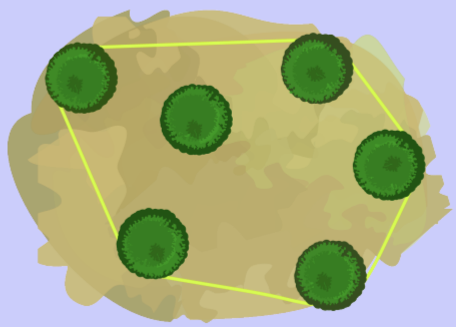
Grade 6 Math Circles
February 28 March 1/2, 2023
BCC/Gauss Prep - Problem Set

BCC Practice Problems

Roped Trees

Joni Beaver uses rope to mark groups of trees. The rope forms a very tight loop so that each tree either touches the rope or is entirely inside the loop. Below is an example where the rope touches exactly 5 trees when viewed from above.



How many trees will the rope touch if the trees are arranged as follows (when viewed from above)?

(A) 4 (B) 5 (C) 6 (D) 7

Solution

We can count the trees touching the rope in the picture below.

So (C) must be the answer.

Robots

Consider these five statements describing the three robots below:

1. Bob and Moe are smiling.
2. Bob, Moe, and Lea each have two legs.
3. Moe has a round head and exactly one leg.
4. All three robots have five fingers.
5. Lea or Bob have their hands raised.

Which of these five statements are true?

(A) 2 and 3 (B) 1 and 3 (C) 1 and 5 (D) None

Solution

We can see that all the robots are smiling so Statement 1 is true. Since Bob's hands are raised, Statement 5 is true. Notice that Statements 2 is not true because Lea only has one leg. Statement 3 is not true because Moe has two legs. Finally, Statement 4 is not true because Moe has less than five fingers. (Lea also has less than five fingers.) So (C) must be the answer.

Balloons

Mark goes to a birthday party. A room at the party is decorated with balloons in rows:

Row 1: 

Row 2: 

Row 3: 

Row 4: 

Mark can't see colours clearly. For him, yellow (C) looks the same as green (A), and blue (D) looks the same as red (B).

Which two rows of balloons look the same to Mark?

(A) Row 1 and Row 4

(B) Row 2 and Row 4

(C) Row 1 and Row 2

(D) Row 1 and Row 3

Solution

If we write out the symbols we get:

Row 1: CBAEBAFAD

Row 2: CBAEABFAD

Row 3: ADCEDAFAB

Row 4: ADCEBBFAC

Then we substitute each C with A because these look the same to Mark:

Row 1: ABAEBAFAD

Row 2: ABAEABFAD

Row 3: ADAEDAFAB

Row 4: ADAEBBFAA

We also substitute each D with B because these look the same to Mark:

Row 1: ABAEBAFAB

Row 2: ABAEABFAB

Row 3: ABAEBAFAB

Row 4: ABAEBBFAA

Now, we can easily see that Row 1 and Row 3 look the same to Mark.

So the answer must be (D)

What to Wear?

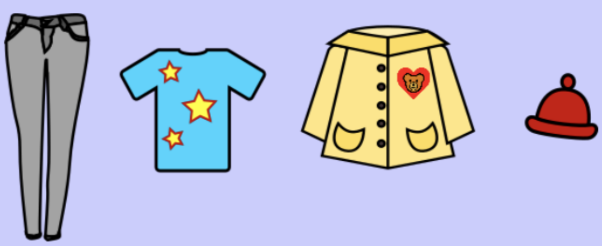
Every morning Maja decides what to wear for the day. She uses the following rules:

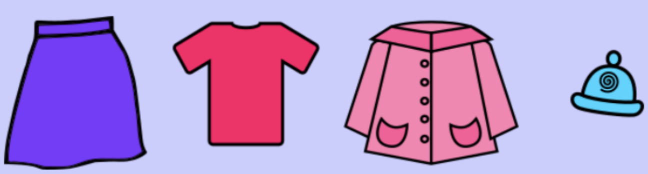
1. If she wears pants, then she wears a T-shirt that is blank or has stars.
2. If she wears a skirt, then she wears a T-shirt with a beaver logo.
3. If she wears a T-shirt that is blank or has stars, then she wears a jacket with a heart.
4. If she wears a jacket with a heart, then she wears a cap with a drawing.

Which of the following combinations can Maja wear?

(A) 

(B) 

(C) 

(D) 

Solution

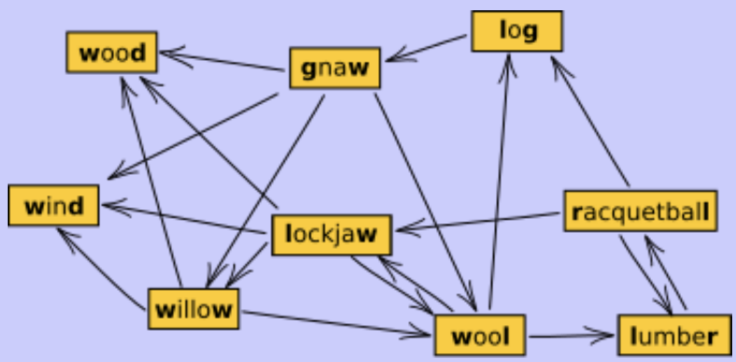
Option B satisfies all four rules. Specifically, rule 1 does not apply because there are no pants in this combination, rule 2 is satisfied because the T-shirt in this combination has a beaver logo on it, rule 3 does not apply for the same reason, and rule 4 does apply because the jacket does not have a heart on it.

Option A violates rule 1. Option C violates rule 4. Option D violates rule 2.

So the answer must be (B).

Longest Word Chain

Beavers play a word chain game. One beaver starts by saying a word. The other beaver must say a different word which begins with the last letter of the previous word. Then the first beaver says another word (which was not said yet) using this same rule, and so on. If a beaver is unable to say a new word, that beaver loses the game. These beavers do not know many words. In fact, they can draw their entire vocabulary like this:



Notice that an arrow out of a word points at the next possible word(s) that can be said.

What is the largest possible number of words that can be said in one game?

(A) 6 (B) 7 (C) 8 (D) 9

Solution

The beavers can use at most 8 words in one game. One example is:

lockjaw-wool-lumber-racquetball-log-gnaw-willow-wood

(Can you find another game of the same length?)

To be sure that 8 is the largest possible number of words, we have to convince ourselves that it is not possible to use all 9 words. Consider the words wood and wind. There is no word beginning with d, so if either of these words is said, it must be the last word of the game. Since there cannot be two words that are said last, it is not possible to use the entire vocabulary of 9 words.

Gauss Practice Problems

1. Erin receives \$3 a day. How many days will it take Erin to receive a total of \$30?

   (A) 8 (B) 12 (C) 14 (D) 27 (E) 10

> **Solution**
> Since $30 \div 3 = 10$, it will take Erin 10 days to receive a total of \$30. So the answer is (E).

2. A class begins at 8:30 a.m. and ends at 9:05 a.m. on the same day. In minutes, what is the length of the class?

   (A) 15 (B) 25 (C) 35 (D) 45 (E) 75

> **Solution**
> There are 30 minutes from 8:30 a.m. to 9:00 a.m. and 5 minutes from 9:00 a.m. to 9:05 a.m. So the length of the class is 35 minutes, and the answer is (C).

3. Chaz gets on the elevator on the eleventh floor. The elevator goes down two floors, then stops. Then the elevator goes down four more floors and Chaz gets off the elevator. On what floor does Chaz get off the elevator?

   (A) 7^{th} floor (B) 9^{th} floor (C) 4^{th} floor (D) 5^{th} floor (E) 6^{th} floor

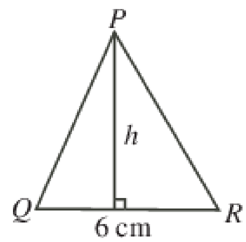
> **Solution**
> $11 - 2 - 4 = 5$, so Chaz got off the 5^{th} floor. The answer is (D).

4. Alex pays \$2.25 to take the bus. Sam pays \$3.00 to take the bus. If they each take the bus 20 times, how much less would Alex pay than Sam in total?

   (A) \$25 (B) \$10 (C) \$15 (D) \$45 (E) \$60

> **Solution**
> Alex pays \$0.75 less than Sam does per trip. So since $\$0.75 \times 20 = \$15$, Alex pays \$15 less than Sam in total. The answer is (C).

5. $\triangle PQR$ has an area of 27 cm and a base measuring 6 cm. What is the height, h, of $\triangle PQR$?



(A) 9 cm (B) 18 cm (C) 4.5 cm (D) 2.25 cm (E) 7 cm

Solution
The formula for the area of a triangle is $A = b \times h \div 2$ (where A represents the area and b represents the base). So isolating for the height, h, and substituting our values for area and base, we get:

$$\begin{aligned} A &= b \times h \div 2 \\ h &= 2A \div b \\ h &= 2(27 \text{ cm}) \div (6 \text{ cm}) \\ h &= 9 \text{ cm} \end{aligned}$$

6. If the mean (average) of five consecutive integers is 21, the smallest of the five integers is
(A) 17 (B) 21 (C) 1 (D) 18 (E) 19

Solution
Since the mean is 21, the five numbers must be 19, 20, 21, 22, and 23. So the smallest of the five integers is 19, and the answer is (E).

7. Karl has 30 birds. Some of his birds are emus and the rest are chickens. Karl hands out 100 treats to his birds. Each emu gets 2 treats and each chicken gets 4 treats. How many chickens does Karl have?
(A) 10 (B) 15 (C) 25 (D) 20 (E) 6

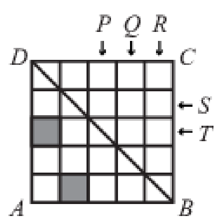
Solution

Let x represent the number of emus that Karl has and y represent the number of chickens he has. Since Karl has a total of 30 birds, we have that $30 = x + y(\star)$. Also, since Karl has 100 treats and gives each emu 2 treats and each chicken 4 treats, we have that $100 = 2x + 4y(\star\star)$. We can rearrange $(\star)$ into $x = 30 - y$ and substitute this into $(\star\star)$. This gives:

$$
\begin{aligned}
100 &= 2x + 4y \\
100 &= 2(30 - y) + 4y \\
100 &= 60 - 2y + 4y \\
40 &= 2y \\
y &= 20
\end{aligned}
$$

So there are 20 chickens, and the answer is (D).

8. In the diagram, which of the following squares should be shaded to make BD a line of symmetry of square $ABCD$?



(A) P and S (B) Q and S (C) P and T (D) Q and T (E) P and R

Solution

Flipping the squares along the line of symmetry, we see that squares P and S should be shaded. So the answer is (A).

9. The mean (average) of the four integers 78, 83, 82, and x is 80. Which one of the following statements is true?

(A) x is 2 greater than the mean

(B) x is 1 less than the mean

(C) x is 2 less than the mean

(D) x is 3 less than the mean

(E) x is equal to the mean

Solution

The formula for the mean of the four numbers is mean $= \frac{78+83+82+x}{4} = 80$. So we have:

$$80 = \frac{78 + 83 + 82 + x}{4}$$
$$320 = 78 + 83 + 82 + x$$
$$x = 77$$

Since 77 is 3 less than 80, the answer is (D).

10. If x, y, and z are positive integers with $xy = 18$, $xz = 3$, and $yz = 6$, what is the value of $x + y + z$?

(A) 6 (B) 10 (C) 25 (D) 11 (E) 8

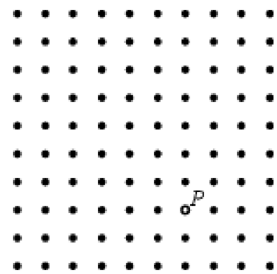
Solution

Since 3 is a prime number, we know from $xz = 3$ that either x and z must be 3 and 1. If we take $x = 3$ and $z = 1$, then we know from $yz = 6$ both $xy = 18$ that $y = 6$, since $3 \times 6 = 18$. So $x + y + z = 3 + 6 + 1 = 10$.

If we take $x = 1$ and $z = 3$, then we know from $yz = 6$ that $y = 2$, since $2 \times 3 = 6$. Then we would get from $xy = 18$ that $x = 9$, since $2 \times 9 = 18$. But this would not work since we already said that $x = 1$.

So the answer is (B).

11. A 10 by 10 grid is created using 100 points, as shown. Point P is given. One of the other 99 points is randomly chosen to be Q. What is the probability that the line segment PQ is vertical or horizontal?



(A) $\frac{2}{11}$ (B) $\frac{1}{5}$ (C) $\frac{1}{10}$ (D) $\frac{4}{25}$ (E) $\frac{5}{33}$

Solution

Line segment PQ is vertical if Q is chosen from the points in the column in which P lies. This column contains 9 points other than P which could be chosen to be Q so that PQ is vertical.

Line segment PQ is horizontal if Q is chosen from the points in the row in which P lies. This row contains 9 points other than P which could be chosen to be Q so that PQ is horizontal. Each of these 9 points is different from the 9 points in the column containing P. Thus, there are $9 + 9 = 18$ points which may be chosen to be Q so that PQ is vertical or horizontal.

Since there are a total of 99 points to choose Q from, the probability that Q is chosen so that PQ is vertical or horizontal is $\frac{18}{99}$ or $\frac{2}{11}$.

So the answer is (A).

12. Celyna bought 300 grams of candy A for \$5.00, and x grams of candy B for \$7.00. She calculated that the average price of all of the candy that she purchased was \$1.50 per 100 grams. What is the value of x?

(A) 525 (B) 600 (C) 500 (D) 450 (E) 900

Solution

Celyna spent \$5.00 on candy A and \$7.00 on candy B, or \$12.00 in total. The average price of all the candy that she purchased was \$1.50 per 100 grams. This means that if Celyna bought 100 grams of candy, she would have spent \$1.50. If she bought 200 grams of candy, she would have spent $2 \times \$1.50 = \$3.00$. How many grams of candy would

Celyna need to buy to spend \$12.00?

Since $8 \times \$1.50 = \$12.00$ (or $\$12.00 \div \$1.50 = 8$), then she would need to buy a total of 800 grams of candy. Celyna bought 300 grams of candy A, and so she must have purchased $800 - 300 = 500$ grams of candy B. The value of x is 500.

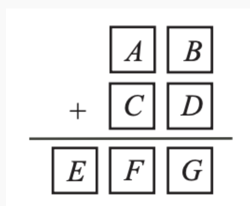
So the answer is (C).

13. In the addition of two 2-digit numbers, each blank space, including those in the answer, is to be filled with one of the digits 0, 1, 2, 3, 4, 5, 6, each used exactly once. The units digit (one's digit) of the sum is

(A) 2 (B) 3 (C) 4 (D) 5 (E) 6

Solution

We label the blank spaces to make them easier to refer to.



Since we are adding two 2-digit numbers, then their sum cannot be 200 or greater, so E must be a 1 if the sum is to have 3 digits.

Where can the digit 0 go?
Since no number can begin with a 0, then neither A nor C can be 0. Since each digit is different, then neither B and D can be 0, otherwise both D and G or B and G would be the same. Therefore, only F or G could be 0.

Since we are adding two 2-digit numbers and getting a number which is at least 100, then $A + C$ must be at least 9. (It could be 9 if there was a "carry" from the sum of the units digits.) This tells us that A and C must be 3 and 6, 4 and 5, 4 and 6, or 5 and 6.

If G was 0, then B and D would have to 4 and 6 in some order. But then the largest that A and C could be would be 3 and 5, which are not among the possibilities above. Therefore, G is not 0, so $F = 0$.

$$\begin{array}{cccc} & & \boxed{A} & \boxed{B} \\ + & & \boxed{C} & \boxed{D} \\ \hline & \boxed{1} & \boxed{0} & \boxed{G} \end{array}$$

So the sum of A and C is either 9 or 10, so A and C are 3 and 6, 4 and 5, or 4 and 6. In any of these cases, the remaining possibilities for B and D are too small to give a carry from the units column to the tens column. So in fact, A and C must add to 10, so A and C are 4 and 6 in some order. Let's try $A = 4$ and $C = 6$.

$$\begin{array}{cccc} & & \boxed{4} & \boxed{B} \\ + & & \boxed{6} & \boxed{D} \\ \hline & \boxed{1} & \boxed{0} & \boxed{G} \end{array}$$

The remaining digits are 2, 3 and 5. To make the addition work, B and D must be 2 and 3 and G must be 5. (We can check that either order for B and D works, and that switching the 4 and 6 will also work.)

So the units digit of the sum must be 5, as in the example

$$\begin{array}{cccc} & & \boxed{4} & \boxed{2} \\ + & & \boxed{6} & \boxed{3} \\ \hline & \boxed{1} & \boxed{0} & \boxed{5} \end{array}$$

So the answer is (D).

(Note that we could have come up with this answer by trial and error instead of this logical procedure.)

14. How many of the five numbers 101, 148, 200, 512, 621 cannot be expressed as the sum of two or more consecutive positive integers?

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

Solution

We begin by showing that each of 101, 148, 200, and 621 can be expressed as the sum of two or more consecutive positive integers.

$$
\begin{aligned}
101 &= 50 + 51 \\
148 &= 15 + 16 + 17 + 18 + 19 + 20 + 21 + 22 \\
200 &= 38 + 39 + 40 + 41 + 42 \\
621 &= 310 + 311
\end{aligned}
$$

We show that 512 cannot be expressed a sum of two or more consecutive positive integers. This will tell us that one of the five numbers in the list cannot be written in the desired way, and so the answer is (B).

Now, 512 cannot be written as the sum of an odd number of consecutive positive integers. Why is this? Suppose that 512 equals the sum of p consecutive positive integers, where $p > 1$ is odd. Since p is odd, then there is a middle integer m in this list of p integers. Since the numbers in the list are equally spaced, then m is the average of the numbers in the list. (For example, the average of the 5 integers 6, 7, 8, 9, 10 is 8.)

But the sum of the integers equals the average of the integers (m) times the number of integers (p). That is, $512 = mp$. Now $512 = 29$ and so does not have any odd divisors larger than 1. Therefore, 512 cannot be written as mp since m and p are positive integers and $p > 1$ is odd. Thus, 512 is not the sum of an odd number of consecutive positive integers.

Further, 512 cannot be written as the sum of an even number of consecutive positive integers. Why is this? Suppose that 512 equals the sum of p consecutive positive integers, where $p > 1$ is even. Since p is even, then there is not a single middle integer m in this list of p integers, but rather two middle integers m and $m + 1$. Since the numbers in the list are equally spaced, then the average of the numbers in the list is the average of m and $m + 1$, or $m + \frac{1}{2}$. (For example, the average of the 6 integers 6, 7, 8, 9, 10, 11 is 8.5.)

But the sum of the integers equals the average of the integers $(m + \frac{1}{2})$ times the number of integers (p). That is, $512 = (m + \frac{1}{2})p$ and so $2(512) = 2(m + \frac{1}{2})p$ or $1024 = (2m + 1)p$. Now $1024 = 2^{10}$ and so does not have any odd divisors larger than 1. Therefore, 1024 cannot be written as $(2m+1)p$ since m and p are positive integers and $2m+1>1$ is odd.

Thus, 512 is not the sum of an even number of consecutive positive integers. Therefore, 512 is not the sum of any number of consecutive positive integers. A similar argument shows that every power of 2 cannot be written as the sum of any number of consecutive positive integers. Returning to the original question, exactly one of the five numbers in the original list cannot be written in the desired way, and so the answer is (B).

15. Two different 2-digit positive integers are called a reversal pair if the position of the digits in the first integer is switched in the second integer. For example, 52 and 25 are a reversal pair. The integer 2015 has the property that it is equal to the product of three different prime numbers, two of which are a reversal pair. Including 2015, how many positive integers less than 10,000 have this same property?

(A) 18 (B) 14 (C) 20 (D) 17 (E) 19

Solution

All 2-digit prime numbers are odd numbers, so to create a reversal pair, both digits of each prime must be odd (so that both the original number and its reversal are odd numbers). We also note that the digit 5 cannot appear in either prime number of the reversal pair since any 2-digit number ending in 5 is not prime. Combining these two facts together leaves only the following list of prime numbers from which to search for reversal pairs: 11, 13, 17, 19, 31, 37, 71, 73, 79, and 97. This allows us to determine that the only reversal pairs are: 13 and 31, 17 and 71, 37 and 73, and 79 and 97. (Note that the reversal of 11 does not produce a different prime number and the reversal of 19 is 91, which is not prime since $7 \times 13 = 91$.)

Given a reversal pair, we must determine the prime numbers (different than each prime of the reversal pair) whose product with the reversal pair is a positive integer less than 10,000. The product of the reversal pair 79 and 97 is $79 \times 97 = 7663$. Since the smallest prime number is 2 and $2 \times 7663 = 15,326$, which is greater than $10,000$, then the reversal pair 79 and 97 gives no possibilities that satisfy the given conditions. We continue in this way, analyzing the other 3 reversal pairs, and summarize our results in the table below.

Prime Number	Product of the Prime Number with the Reversal Pair			
	13 and 31	17 and 71	37 and 73	79 and 97
2	$2 \times 13 \times 31 = 806$	$2 \times 17 \times 71 = 2414$	$2 \times 37 \times 73 = 5402$	greater than 10,000
3	$3 \times 13 \times 31 = 1209$	$3 \times 17 \times 71 = 3621$	$3 \times 37 \times 73 = 8103$	
5	$5 \times 13 \times 31 = 2015$	$5 \times 17 \times 71 = 6035$	greater than 10 000	
7	$7 \times 13 \times 31 = 2821$	$7 \times 17 \times 71 = 8449$		
11	$11 \times 13 \times 31 = 4433$	greater than 10 000		
13	can't use 13 twice			
17	$17 \times 13 \times 31 = 6851$			
19	$19 \times 13 \times 31 = 7657$			
23	$23 \times 13 \times 31 = 9269$			
29	greater than 10 000			
Total	8	4	2	0

In any column, once we obtain a product that is greater than 10,000, we may stop evaluating subsequent products since they use a larger prime number and thus will exceed the previous product. In total, there are $8 + 4 + 2 = 14$ positive integers less than 10,000 which have the required property.

More Practice

To access any previous contest problems and solutions, go to:
https://www.cemc.uwaterloo.ca/contests/past_contests.html